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An Introduction to de Rham Cohomology

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Abstract

This paper provides an expository introduction to de Rham cohomology. The de Rham cohomology is a tool in both algebraic and differential topology that connects the topological structure of smooth manifolds to computations relating to differential forms. This paper begins by developing foundational knowledge of smooth manifolds and differential forms, including their algebraic structures, pullbacks, and the exterior derivative. Building upon these concepts leads to a discussion of the integration of forms on manifolds and the significance of closed and exact forms, including a statement and interpretation of Stokes' theorem. The core of this paper is dedicated to defining the de Rham cohomology groups and demonstrating the use of tools in computing cohomology groups. Finally, some extensions and applications of the de Rham cohomology group are explored. Overall, this paper aims to make de Rham cohomology and its applications accessible to readers through a clear, structured, and intuitive way.

Introduction

De Rham cohomology is named after the Swiss mathematician Georges de Rham, who first introduced the theory in the early 20th century. The theory can be traced back to de Rham's doctoral thesis in 1931, where he studied the relationship between differential forms and the topology of differentiable manifolds. Although terminology and complete formalism were not yet in place, de Rham effectively used what would later be recognized as cohomology groups to establish foundational results. The formal definition of de Rham cohomology appeared in his 1938 paper *Sur l'analyse situs des variétés à n dimensions*. De Rham cohomology has since become a central tool in modern geometry and topology. It laid the foundations for the development of other cohomological theories, including Čech cohomology, sheaf cohomology, and Hodge theory.

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Motivations

Generally, in mathematics, a central goal is to identify invariants, which are quantities or structures that remain unchanged under certain transformations, because they allow classification of objects up to equivalence. In topology, invariants characterize spaces in ways that are independent of specific representations or coordinate choices. These invariants can take various forms, such as the Euler characteristic, which is an integer-valued invariant, or orientability, which is a binary invariant. The fundamental group, another key topological invariant, encodes information about the space's loops up to continuous deformation. At a more advanced level, cohomology theories provide algebraic invariants that detect features like connectedness and the number of holes in various dimensions. Among these, de Rham cohomology stands out for smooth manifolds. Although defined using differential forms and the exterior derivative, it produces vector spaces that are invariant under smooth homotopy equivalence. This makes de Rham cohomology useful for translating geometric and analytic information into topological classification.
Furthermore, de Rham cohomology has deep implications within calculus. The traditional fundamental theorem of calculus gives that if an integral begins and ends at the same place, i.e. the integral is taken over a closed loop, then its value is zero. Extending this idea to higher dimensions, it is expected that the integral over a closed loop should vanish. In Euclidean space, this expectation largely holds. However, it fails when dealing with more general manifolds. An example arises when integrating around a loop about zero in the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Here, a function that locally appears to have a well-defined derivative, like the multi-valued angle function θ, exhibits globally non-trivial behavior. Integrating the differential $d\theta$ around this loop yields a nonzero result, showing a relationship between integration and the "holes" in the manifold. This failure of the fundamental theorem in higher dimensions can be rigorized and measured by de Rham cohomology. It transforms the analytic behavior of differential forms into algebraic invariants that reflect the topology of the space. The study of de Rham cohomology is thus a central tool in connecting differential calculus on manifolds with their topological properties.

Smooth Manifolds

Basic Topological Notions

Before progressing further, it is useful to recall a few basic topological definitions that play an important role in the study of smooth manifolds.

Definition. A topological space is **compact** if every open cover has finite subcover.

Intuitively, this means the space can be "covered" using only finitely many of the open sets. For example, the closed interval $[0, 1]$ in $\mathbb{R}$ is compact, whereas $\mathbb{R}$ itself is not.

Definition. A topological space is **connected** if it cannot be written as the union of two nonempty disjoint open sets.

This is the same as being path connected for manifolds, which means there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$ for all $p, q \in X$.

Charts and Smoothness

With these foundational topological concepts in place, the discussion now turns to the structure of smooth manifolds. Generally speaking, a manifold is a topological space that resembles Euclidean space locally. In particular, locally Euclidean means that for all $x \in X$, where X is a n-manifold, there exists an open neighborhood $U \ni x$ such that there exists a homeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$, where V is open.

Definition. A **n-dimensional manifold** is a Hausdorff space that is second countable and locally Euclidean of dimension n.

Every topological manifold has a specific well-defined dimension, so spaces of mixed dimension are not manifolds. Furthermore, the dimension of a topological manifold is a topological invariant, which will be shown later. To move from the purely topological structure of manifolds to a richer differential structure, one must introduce the notion of smoothness. This allows for the application of calculus on manifolds by ensuring that functions and coordinate changes behave well with respect to differentiation. This begins by defining smoothness in the familiar setting of Euclidean spaces.

Definition. Given open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, a function $f : U \to V$ is said to be **smooth** (or infinitely differentiable as an element of $C^\infty(U)$)

if every component has continuous partial derivatives of all orders. If f is bijective and also has a smooth inverse map, it is called a **diffeomorphism**.

To define smoothness on a manifold, the structure must permit local coordinate descriptions that map to Euclidean space, where differentiability is well-established. These local representations, known as charts, must align in a smooth manner on overlapping domains. This requirement leads to the concept of transition maps and their smooth compatibility, which are essential in establishing a smooth structure on a manifold. The definitions that follow formalize these notions.

Definition. A **chart** on an n-manifold X is a pair (U, φ) where $U \subseteq X$ open and $\varphi : U \to V \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset in $\mathbb{R}^n$.

Definition. Given charts (U, φ) and (V, ψ) such that $U \cap V \neq \varnothing$, the **transition map** is given by $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. Two charts (U, φ) and (V, ψ) are **smoothly compatible** if $U \cap V = \varnothing$ or the transition map is a diffeomorphism.

Roughly speaking, the transition map is the change of coordinates between charts.

Definition. An **atlas** $\mathcal{A}$ is a collection of charts that form an open cover of the manifold. $\mathcal{A}$ is a **smooth atlas** if any two charts are smoothly compatible with each other.

Given these definitions, a formal definition of smooth manifolds appears.

Definition. A **smooth structure** on manifold X is a maximal atlas.

This means that all possible compatible combinations are in the collection. However, not all manifolds have a smooth structure. Therefore, smooth manifolds can be defined as follows.

Definition. A **smooth manifold** is a pair $(X, \mathcal{A})$ where X is a topological manifold and $\mathcal{A}$ is a smooth structure on X.

Generally, when smooth structures are understood, they are omitted, and it is just noted that "X is a smooth manifold." Occasionally, there are spaces that would be smooth manifolds, except they have a "boundary" in some sense. Similar notions of smoothness can be defined for these spaces using the following definitions.

Definition. A **n-dimensional manifold with boundary** is a Hausdorff space that is second-countable such that every point has a neighborhoods that is homeomorphic to an open subset in $\mathbb{R}^n$ or a relatively open subsets of the closed n-dimensional upper half-space $\mathbb{H}^n \subseteq \mathbb{R}$.

Definition. A **chart** on an n-manifold with boundary M is a pair (U, φ) where $U \subseteq X$ open and $\varphi : U \to V \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset in $\mathbb{R}^n$ or $\mathbb{H}^n$.

While this definition is extremely similar to that of n-manifolds without boundary, a distinction can be made between certain charts of M that allows interior and boundary points to be distinguished.

Definition. An **interior chart** of M is a chart (U, φ) such that $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

Definition. A **boundary chart** of M is a chart (U, φ) such that $\varphi(U)$ is an open subset of $\mathbb{H}^n$ and $\varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$.

Definition. A **interior point** of M is a point $p \in M$ such that it is in the domain of some interior chart.

Definition. A **boundary point** of M is a point $p \in M$ such that it is in the domain of some boundary chart that sends p to $\partial\mathbb{H}^n$.

Finally, there is a rigorous distinction between the boundary and interior of a manifold.

Definition. The **interior** of M is the set of all its interior points, denoted by $\mathrm{Int}M$.

Definition. The **boundary** of M is the set of all its boundary points and is denoted by ∂M.

The definitions for smooth structure and hence smooth manifolds can be extended using the same concept of smoothness to smooth manifolds with boundary. Every n-manifold without boundary can also be considered an n-manifold with boundary, where its boundary is empty. Every smooth manifold, then, is automatically a smooth manifold with boundary. For the purposes of this text, all manifolds, maps, and functions can be assumed to be smooth unless otherwise noted.

Tangent Spaces

In order to define calculus on manifolds, there needs to be a notion of tangent vectors and spaces. To do this, consider paths that go through a point take the velocities and direction of the path to get tangent vectors. Taking the velocities of all the paths gives all of the tangent vectors. A more rigorous definition is as follows.

Definition. A **tangent vector** is a triple $((U, \varphi), \vec{v})$ where $\vec{v} \in \mathbb{R}$ and (U, φ) is a chart around X.Two vectors $((U, \varphi), \vec{v})$ and $((W, \psi), \vec{z})$ are equivalent if $D(\varphi \circ \psi^{-1})_{\psi(X)}\vec{z} = \vec{v}$.

This means that taking the derivative of the change of coordinates (transition function) and applying it maps one vector to another.

Definition. The **tangent space** to X at $x \in X$, written T_pX, is the space of all tangent vectors.

If a chart is fixed, then there is exactly one tangent vector for each path. So, any smooth coordinate chart gives a basis for each tangent space.
For a general real vector space V, such as the tangent space, we can define a dual space.

Definition. The **dual space** V^* of vector space V is a real vector space of linear maps $f : V \to \mathbb{R}$.

Furthermore, $\dim V = \dim V^*$ as long as the values are finite. Now, fix a basis $v_1, ..., v_k$ for V. Let $x^i \in V^*$ be the map which picks out the ith Then, any vector $v \in V$ can be written as $v = x^1(v)v_1 + ... + x^k(v)v_k$. Note that the functions $x^i : V \to \mathbb{R}$ are linear and form a basis for V^*. Finally, for a linear map $\mathcal{A} : \mathcal{V} \to \mathcal{W}$, there exists a linear dual map.

Definition. A **dual map** $\mathcal{A}^* : \mathcal{W}^* \to \mathcal{V}^*$ is defined by $\mathcal{A}^*(f) = f \circ \mathcal{A}$.

In particular, the matrix of $\mathcal{A}^* : (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$ is the transpose $\mathcal{A}^T$ and $\mathcal{A}^* \circ \mathcal{B}^* = (\mathcal{B} \circ \mathcal{A})^*$. Finally, applying this to the tangent space results in cotangent spaces.

Definition. The **cotangent space** T_p^*X of manifold X at point $p \in X$ is the dual space of the tangent space T_pX.

In other words, T_p^*X is the real vector space of all linear maps $T_pX \to \mathbb{R}$. Note that the dimension of the cotangent space is the same as the dimension of the tangent space.

Definition. The **cotangent bundle** T^*X, is the vector bundle of the cotangent spaces at each point.

Differential Forms

Finally, there is enough structure to define differential forms on manifolds.

Differential 1-forms

Definition. A **differential 1-form** ω on a manifold X is a smooth section of the cotangent bundle T^*X.

In other words, a 1-form $\omega \in \Omega^1(X)$ is a smooth function from $X \to T^*X$ that assigns a function $\omega_p \in T_p^*X$ to every point $p \in X$. For example, take an open set $U \subset \mathbb{R}^n$. Then, a 1-form $\omega \in \Omega^1(U)$ is a function $\omega : U \to T_p^*\mathbb{R}^n = (\mathbb{R}^n)^*$ since the tangent space of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

Example. Consider the directional derivative $(df)_p$ of $f \in C^\infty(U)$ for some $p \in U$. Then, for some vector v in the tangent space

$$(df)_p(v) = \left.\frac{d}{dt}\right|_{t=0} f(p+tv)$$

Using the chain rule gives

$$(df)_p(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) x^i(v)$$

This shows that $p \mapsto (df)_p$ is a map from $U \to (\mathbb{R}^n)^*$. The i-th coefficient of $(df)_p$ with the standard dual basis is $\frac{\partial f}{\partial x^i}(p)$ which is a smooth function of p. So, $df \in \Omega^1(U)$. A special case of this is example is taking $f = x^i$. Then, $(dx^i)_p(v) = x^i(v)$.

Lemma 0.0.1. If $\omega \in \Omega^1(U)$, then there exist functions $Q_i \in C^\infty(U)$ such that

$$\omega = \sum_{i=1}^{n} Q_i dx^i.$$

Proof. The dx^i form a basis for 1-forms, so $Q_i(p)$ can be taken to be the i-th coordinate of $\omega_p \in (\mathbb{R}^n)^*$. Then, $Q_i \in C^\infty(U)$ by smoothness of ω. □

It is also important to consider the derivative of a function along a curve for future notions of integration on curves.

Proposition. Suppose X is a manifold with or without boundary, $\gamma : J \to X$ is a smooth curve, and $f : X \to \mathbb{R}$ is a function. Then the derivative of

the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

General k-forms

The concept of differential 1-forms can be generalized to higher dimensions by introducing differential k-forms. Intuitively, a k-form is an object that can be integrated over a k-dimensional oriented surface. More formally, k-forms are alternating, multilinear maps acting on k-tuples of tangent vectors.

Definition. Let V be a real vector space. The **k-th exterior power** of V, denoted $\Lambda^k V^*$, is the vector space of all alternating k-linear maps from V^k to $\mathbb{R}$.

Definition. Let X be a manifold. A **differential k-form** ω on X is a smooth section of the vector bundle $\Lambda^k T^*X$. That is, for each $p \in X$, ω_p is an element of $\Lambda^k T_p^* X$, the k-th exterior power of the cotangent space at p.

Note that the set of differential k-forms on a manifold X is denoted by $\Omega^k(X)$.

Definition. The **wedge product** $\alpha \wedge \beta : V \times V \to \mathbb{R}$ is defined by

$$(\alpha \wedge \beta)(u, v) := \det \begin{pmatrix} \alpha(u) & \beta(v) \\ \beta(u) & \beta(v) \end{pmatrix}$$

Proposition. Given $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^\ell V^*$, the wedge product $\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$ is the alternating multilinear form defined by the following properties:

1. Bilinearity: $(a\alpha + b\alpha') \wedge \beta = a(\alpha \wedge \beta) + b(\alpha' \wedge \beta)$.
2. Antisymmetry: $\alpha \wedge \beta = (-1)^{k \cdot \ell} \beta \wedge \alpha$

The wedge product can be generalized to any number of terms with multilinearity, antisymmetric, and associative properties.

The wedge product allows for the construction of higher-degree forms. In local coordinates, any k-form $\omega \in \Omega^k(X)$ can be expressed as a linear combination of wedge products of basis 1-forms. That is,

$$\omega = \sum_{1 \le i_1 < \cdots < i_k} f_{i_1 \ldots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where each $f_{i_1 \dots i_k} \in C^\infty(X)$ is a smooth function. The set $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}_{1 \le i_1 < \cdots < i_k}$ forms a local basis for $\Omega^k(X)$.

Exterior Derivative

To define a cohomology theory from differential forms, it is necessary to introduce a derivative that increases form degree and generalizes the usual differential of functions. This leads to the concept of the exterior derivative, which extends the differential df of a smooth function f to higher-degree differential forms. The exterior derivative is defined locally and acts by taking partial derivatives in a way consistent with the structure of differential forms.

Definition. The **exterior derivative** is an operator

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

defined by the following:

1. For a 0-form $f \in \Omega^0(X)$, df is the usual differential of f, which in local coordinates $(x^1, \dots, x^n)$ is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

2. For a k-form $\omega = \sum_I f_I dx^I \in \Omega^k(X)$ in local coordinates, where $I = (i_1, \dots, i_k)$ is a multi-index and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$,

$$d\omega = \sum_I df_I \wedge dx^I = \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x^j} dx^j \wedge dx^I$$

Proposition. Let X be a smooth manifold. The exterior derivative satisfies the following properties:

1. $d(\alpha + \beta) = d\alpha + d\beta$
2. If $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^\ell(X)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

3. $d \circ d = 0$, i.e., $d^2 = 0$

The exterior derivative allows the definition of two fundamental classes of differential forms.

Definition. A differential form $\omega \in \Omega^k(X)$ is called **closed** if $d\omega = 0$.

Definition. A differential form $\omega \in \Omega^k(X)$ is called **exact** if there exists a $(k-1)$-form $\eta \in \Omega^{k-1}(X)$ such that $\omega = d\eta$.

Since $d^2 = 0$, every exact form is necessarily closed. However, the converse is not always true: not every closed form is exact.

One important feature of the differential is the characterization of functions with vanishing differentials.

Proposition. If f is a smooth function on a manifold X with or without boundary, then $df = 0$ if and only if f is constant on each component of X.

Pullbacks of Forms

To evaluate or compare differential forms on different manifolds, it is essential to transfer forms across smooth maps. The operation that enables this is called the pullback.

Definition. Let $F : X \to Y$ be a smooth map between smooth manifolds. Then for each $k \geq 0$, the **pullback** of a differential k-form $\omega \in \Omega^k(Y)$ by F is a k-form $F^*\omega \in \Omega^k(X)$ defined so that for all $p \in X$ and tangent vectors $v_1, \ldots, v_k \in T_pX$:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k)),$$

where $dF_p : T_pX \to T_{F(p)}Y$ is the differential of F at p.

This definition has a specific expression in coordinates when pulling back a 1-form. Suppose $F : V \subseteq \mathbb{R}^k \to U \subseteq \mathbb{R}^n$ is a map, and suppose a 1-form on U is given by

$$\omega = \sum_{i=1}^{n} Q_i dx^i \in \Omega^1(U).$$

Then, the pullback $F^*\omega \in \Omega^1(V)$ is given by

$$(F^*\omega)_p := (dF_p)^*\omega_{F(p)}.$$

More explicitly, where V is represented with coordinates of y_i, observe that

$$(F^*\omega)_p = \omega_{F(p)} \circ dF_p = \sum_{i=1}^{n}\sum_{j=1}^{k} Q_i(F(p))\frac{\partial F^i}{\partial y^j}(p)(dy^j)_p.$$

Example. Let $F : \mathbb{R} \to \mathbb{R}^2$ be defined by $F(t) = (\cos t, \sin t)$ and let $\omega = xdy - ydx \in \Omega^1(\mathbb{R}^2)$. Then:

$$F^*\omega = \cos t d(\sin t) - \sin t d(\cos t) = \cos t \cos t dt + \sin t \sin t dt = dt.$$

The pullback preserves wedge products and commutes with the exterior derivative:

Proposition. $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta, \quad F^*(d\omega) = d(F^*\omega).$

This compatibility makes the pullback essential for defining integrals over parametrized domains.

Integration of Differential Forms

Line Integrals

Let $\omega \in \Omega^1(U)$ be a differential 1-form on an open subset $U \subseteq \mathbb{R}^n$, and let $\gamma : I \to U$ be a smooth parametrized curve, where $I \subset \mathbb{R}$ is an interval. Then the integral of ω along γ is defined by

$$\int_\gamma \omega := \int_I \omega_{\gamma(t)}(\gamma'(t))dt.$$

While it appears to involve $\omega_{\gamma(t)}(\gamma'(t))dt$ as a 1-form on I, this is technically the pullback $\gamma^*\omega$ evaluated on $\frac{d}{dt}$. In other words, we can equivalently write:

$$\int_\gamma \omega = \int_I \gamma^*\omega.$$

Therefore, the pullback of a 1-form under a curve γ gives a 1-form on the interval I, and the line integral is the integral of this pullback. In coordinates, suppose $\omega = \sum_{i=1}^n Q_i dx^i$ and $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Then:

$$\gamma^*\omega = \sum_{i=1}^{n} Q_i(\gamma(t))(\gamma^i)'(t)dt$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} Q_i(\gamma(t)) d(\gamma^i(t)) \\
&= \sum_{i=1}^{n} Q_i(\gamma(t)) \frac{d\gamma^i}{dt}(t) dt \\
&= \omega_{\gamma(t)}(\gamma'(t)) dt.
\end{aligned}
$$

There is now enough information about the line integral to show that it is invariant under reparametization.

Theorem. Let $\sigma : J \to I$ be a diffeomorphism with $\sigma'(t) > 0$, and let $\widetilde{\gamma} = \gamma \circ \sigma$. Then

$$\int_{\widetilde{\gamma}} \omega = \int_{\gamma} \omega.$$

Proof.

$$
\begin{aligned}
\int_{\widetilde{\gamma}} \omega &= \int_J \omega_{\gamma(\sigma(t))}(\gamma'(\sigma(t)))\sigma'(t) dt \\
&= \int_J \omega_{\gamma(\sigma(t))}(\gamma'(\sigma(t))) d\sigma(t) = \int_I \omega_{\gamma(s)}(\gamma'(s)) ds = \int_{\gamma} \omega.
\end{aligned}
$$

□

The proof uses the change of variables formula for Riemann integrals and the chain rule. Note that if $\sigma'(t) < 0$, the same computation holds but introduces a minus sign, reflecting the orientation reversal. Now, the fundamental theorem of calculus can be expanded to the fundamental theorem of line integrals.

Theorem. Let $f \in C^\infty(U)$ and $\gamma : [a, b] \to U$ be a smooth curve. Then

$$\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a)).$$

Proof.

$$\int_{\gamma} df = \int_a^b df_{\gamma(t)}(\gamma'(t)) dt = \int_a^b \frac{d}{dt} f(\gamma(t)) dt = f(\gamma(b)) - f(\gamma(a)).$$

□

This allows for the exploration of the previously stated mismatch between closed and exact forms.

Corollary. Let $\omega \in \Omega^1(U)$, and if there exist a smooth closed curve $\gamma : [a, b] \to U$ with $\gamma(a) = \gamma(b)$ such that

$$\int_\gamma \omega \neq 0.$$

Then there does not exist a function $f \in C^\infty(U)$ such that $\omega = df$.

This result means that if a closed curve has nonzero integral under a 1-form, then that form cannot be exact. One of the most fundamental examples demonstrating this arises from the punctured plane $\mathbb{R}^2 \setminus \{0\}$.

Example. Define the 1-form

$$\omega_{\text{ang}} = \frac{-x^2 dx^1 + x^1 dx^2}{(x^1)^2 + (x^2)^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\}).$$

Consider the closed curve $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Then the pullback of ω_{ang} along γ is:

$$\gamma^* \omega_{\text{ang}} = (\cos^2 t + \sin^2 t) dt = dt,$$

so

$$\int_\gamma \omega_{\text{ang}} = \int_0^{2\pi} dt = 2\pi \neq 0.$$

Thus, ω_{ang} is a closed 1-form on $\mathbb{R}^2 \setminus \{0\}$ that is not exact because there is no $f \in \mathbb{R}^2 \setminus \{0\}$ such that $\omega_{\text{ang}} = df$.

Integration of k-forms

The integration of differential forms generalizes the notion of line integrals to higher dimensions. This requires an understanding of orientation, partitions of unity, in particular compact support, and the behavior of forms under smooth maps and charts. Thus, the general ideas of compact support and orientation must first be established.

Definition. Let f be a function with domain X, a topological space. The **support** $\operatorname{supp} f$ is the closure of the points where f is nonzero ($\operatorname{supp} f = \overline{\{p \in X | f(p) \neq 0\}}$).

Definition. A **compactly supported** function is a function f where $\operatorname{supp} f \subset X$ is compact.

Proposition. Let X be a manifold with or without boundary, and $\{U_\alpha\}_{\alpha\in A}$ is an indexed open cover of X. Then, there is a family $\{\psi_\alpha\}_{\alpha\in A}$ of continuous functions $\psi_\alpha : X \to \mathbb{R}$ such that

1. $0 \leq \psi_\alpha \leq 1$ for all $x \in X$ and $\alpha \in A$
2. $\operatorname{supp}\psi_\alpha \subset U_\alpha$ for all $\alpha \in A$
3. $\sum_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in X$
4. The family of supports $\{\operatorname{supp}\psi_\alpha\}_{\alpha\in A}$ is locally finite. Such a family of functions is a **partition of unity** subordinate to $\{U_\alpha\}_{\alpha\in A}$.

Definition. A map $f : X \to Y$ between manifolds is **orientation-preserving** if the derivative map $df_p : T_pX \to T_pY$ satisfies $\det df_p > 0$ for some $p \in X$. Otherwise, f is **orientation-reversing**.

Definition. A chart (U, φ) for an n-manifold X is **positively oriented** is $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is orientation preserving.

Now, there is enough background to extrapolate the line integral to higher dimensions and define some interesting properties. Let $D \subseteq \mathbb{R}^k$ be open, and suppose $\omega = f dx^1 \wedge \cdots \wedge dx^k$ is a continuous k-form on D, where $f : D \to \mathbb{R}$ is continuous. Then the integral of ω over D is defined as

$$\int_D \omega := \int_D f(x) dx^1 \dots dx^k.$$

This is simply the standard multivariable integral of f over D with respect to the volume form. If $\omega \in \Omega^k_c(U)$ is a compactly supported k-form on an open subset $U \subseteq \mathbb{R}^n$ or the upper half-space $\mathbb{H}^n$, and if $U \subseteq D$ for some open set $D \subseteq \mathbb{R}^n$, then

$$\int_U \omega := \int_D \widetilde{\omega},$$

where $\widetilde{\omega}$ is any compactly supported extension of ω to D. The value of the integral does not depend on the choice of extension.

Proposition. Let D, E be open domains of integration on $\mathbb{R}^k$ or $\mathbb{H}^k$ $F : \overline{D} \to \overline{E}$ is a map that restricts to a diffeomorphism from D onto E. If ω is a compactly supported k-form on E, then

$$\int_E \omega = \begin{cases} \int_D F^*\omega & \text{if } F \text{ is orientation-preserving,} \\ -\int_D F^*\omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

This result is the generalization of the substitution rule from multivariable calculus.

Lemma 0.0.2. Let $U \subseteq \mathbb{R}^k$ (or $\mathbb{H}^k$) be open and let $E \subseteq U$ be compact. Then there is an open set D such that $E \subseteq D \Subset U$ and D is diffeomorphic to an open subset of $\mathbb{R}^k$ (or $\mathbb{H}^k$).

This lemma ensures that locally there is always a coordinate neighborhood in which to perform integration.

Proposition. Let (U, φ) be a coordinate chart with $\omega \in \Omega_c^k(U)$. Then define the integral via the chart as

$$\int_U \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

Since ω has compact support in U, this integral is well-defined and independent of the choice of chart, provided it contains the support.

Proposition. Let $\omega \in \Omega_c^k(X)$ be a compactly supported k-form on a manifold X. Choose a finite open cover $\{U_\alpha\}$ of $\operatorname{supp}\omega$ and a smooth partition of unity $\{\psi_\alpha\}$ subordinate to it. Then,

$$\int_X \omega := \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega.$$

This construction is independent of the choice of open cover and partition of unity, and it allows integration of forms over general manifolds.

prop*erties of Integration

Let X be an oriented n-dimensional manifold, and let $\omega, \eta \in \Omega_c^n(X)$ be compactly supported. Then the following properties hold:

1. For all $a, b \in \mathbb{R}$ $\int_X (a\omega + b\eta) = a \int_X \omega + b \int_X \eta$.
2. If $-X$ denotes the same manifold with the opposite orientation, then $\int_{-X} \omega = -\int_X \omega$
3. If ω is a positively oriented on X, then $\int_X \omega > 0$
4. Let $F : X \to Y$ be a diffeomorphism.
   Then $\int_Y \omega = \begin{cases} \int_X F^*\omega & \text{if } F \text{ is orientation-preserving,} \\ -\int_X F^*\omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$

Finally, the following proposition gives a method to integrate over different parameterizations.

Proposition. Suppose X is an oriented n-manifold and $\omega \in \Omega_c^n(X)$ is compactly supported. Let $\{D_i\}_{i=1}^K$ be open sets in $\mathbb{R}^n$ and $F_i : \overline{D_i} \to X$ be smooth maps satisfying:

1. Each F_i is a diffeomorphism from D_i onto an open subset $\omega_i \subseteq X$.
2. Each F_i is orientation-preserving.
3. The supports $\{\omega_i\}$ cover $\text{supp}(\omega)$.

Then the integral over X may be computed as

$$\int_X \omega = \sum_{i=1}^K \int_{D_i} F_i^* \omega.$$

Stokes' Theorem

Stokes' Theorem is one of the central results in differential geometry. It generalizes several classical theorems from vector calculus, including the fundamental theorem of calculus, Green's theorem, the divergence theorem, and the classical Stokes' theorem for surfaces in $\mathbb{R}^3$. The general version of Stokes' Theorem applies to differential forms on oriented manifolds with boundary.

Theorem. Let X be a smooth, oriented n-dimensional manifold with boundary ∂X, and let $\omega \in \Omega_c^{n-1}(X)$ be a compactly supported smooth $(n-1)$-form on X. Then:

$$\int_X d\omega = \int_{\partial X} \omega.$$

This relates the integral of the exterior derivative $d\omega$ over X to the integral of ω over the boundary ∂X. The theorem is coordinate-free and holds on any smooth oriented manifold with boundary, not just in Euclidean space. The assumption that ω is compactly supported ensures that the integrals are well-defined. The Stokes' Theorem gives the integral of closed forms and exact forms.

Corollary. If X is a compact oriented manifold without boundary, then the integral over every exact form over X is zero:

$$\int_X dw = 0 \text{ if } \partial X = 0$$

Corollary. If X is a compact oriented manifold with boundary, and ω is a closed form on X, then the integral of ω over ∂X is zero:

$$\int_{\partial X} \omega = 0 \text{ if } d\omega = 0 \text{ on } X$$

Corollary. If X is a manifold with or without boundary, $S \subseteq X$ is an oriented compact k-dimensional submanifold without boundary, and ω is a closed k-form on X, if $\int_S \omega \neq 0$, then:

1. ω is not exact on X
2. S is ot the boundary of an oriented compact smooth submanifold with boundary in X

Special Cases

1. If $X = [a, b] \subset \mathbb{R}$, a 1-dimensional manifold, and $\omega = f \in \Omega^0(X)$ is a smooth function, then Stokes' Theorem reduces to the fundamental theorem of calculus:

$$\int_{[a,b]} df = \int_a^b f'(x)dx = f(b) - f(a).$$

2. If $X \subset \mathbb{R}^2$ is a region with boundary and $\omega = Pdx+Qdy$, then Stokes' Theorem gives Green's Theorem:

$$\int_X \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_{\partial X} Pdx + Qdy.$$

3. If $X \subset \mathbb{R}^3$ is a solid region and ω is a 2-form corresponding to a vector field F, then Stokes' Theorem recovers the Divergence Theorem:

$$\int_X (\nabla \cdot F)dV = \int_X (\nabla F)dx_1 \wedge dx_2 \wedge dx_3$$

$$= \int_{\partial X} F_1 dx_2 \wedge dx_3 + F_2 dx_3 \wedge dx_1 + F_3 dx_1 \wedge dx_2 = \int_{\partial X} F \cdot \vec{n} dS.$$

4. If X is a surface with boundary and ω corresponds to a 1-form arising from a vector field, Stokes' Theorem becomes the classical version:

$$\int_X (\nabla \times F) \cdot d\vec{S} = \int_{\partial X} F \cdot d\vec{r}.$$

de Rham Cohomology

Previously, it was shown that not every closed form is exact. For example, a 1-form on the punctured plane that is closed but not exact was given. For higher degrees, whether closed forms are exact depend on topological properties of the manifold, specifically the "holes" of such higher dimension manifolds. Quantifying the difference between closed and exact forms on a manifold lead to invariants known as the de Rham cohomology groups.

de Rham Cohomology Groups

If X is a smooth manifold with or without boundary, and k is a nonnegative integer, since the exterior derivative $d : \Omega^k(X) \to \Omega^{k+1}(X)$ is linear, its kernel and image are linear subspaces. Then, using the kernel and image, the following definitions can be made.

Definition.

$$\mathcal{Z}^k(X) = \ker(d : \Omega^k(X) \to \Omega^{k+1}(X)) = \{\text{closed } k\text{-forms on } X\}$$

Definition.

$$\mathcal{B}^k(X) = \text{im}(d : \Omega^{k-1}(X) \to \Omega^k(X)) = \{\text{exact } k\text{-forms on } X\}$$

For $k < 0$ or $k > n = \dim X$, $\Omega^k(X)$ is thought of as the zero vector space. This is because $\Omega^k(X)$ is the space of smooth differential k-forms on X. However, when $k > n$, there are no nonzero k-forms, since locally k-forms are expressed as wedge products of k differentials dx^i, and there are only n independent directions on an n-dimensional manifold. So, $\mathcal{B}^0(X) = 0$ and $\mathcal{Z}^n(X) = \Omega^n(X)$.
Since every exact form is closed, $\mathcal{B}^k(X) \subseteq \mathcal{Z}^k(X)$. So, the de Rham cohomology can be defined as the difference between the two spaces.

Definition. **The kth de Rham group** of X is the quotient vector space

$$H^k_{dR}(X) = \frac{\mathcal{Z}^k(X)}{\mathcal{B}^k(X)}$$

For the purposes of this paper, since no other homology groups are being discussed, the de rham cohomology group will be denoted as simply $H^k(X)$. The main aspect of the de Rham cohomology groups that makes it extremely useful is that it is homotopy invariant. In particular, homotopy equivalent manifolds have isomorphic de Rham groups.

Proposition. If X and Y are homotopy equivalent manifolds with or without boundary, then $H^k(X) \cong H^k(Y)$ for all k. The isomorphisms are induced by any smooth homotopy equivalence $F : X \to Y$.

Basic Cohomology Examples

Example. $H^0(X)$. Consider the 0th de Rham group of a manifold X. This is always a measure of the number of connected components in the manifold. In other words,

$$H^0(X) = \mathbb{R}^{\#\text{connected components}}$$

This is because 0-forms are simply smooth functions, and by Proposition 3.3.4, a 0-form is closed if and only if it is locally constant. Therefore, a closed 0-form must be constant on each connected component of X. Moreover, there are no exact 0-forms, since exactness would require the existence of a (-1)-form, which does not exist. Consequently, for each connected component of X, the value of a closed 0-form can be freely chosen in $\mathbb{R}$, and there is no further identification or quotienting. Hence, the space of closed 0-forms modulo exact forms is isomorphic to $\mathbb{R}^k$, where k is the number of connected components of X.

Example. $H^k(X)$ where $\dim X = n < k$. If the degree of the de Rham group is larger than the dimension of the manifold, then it is zero. In other words,

$$H^k(X) = 0 \text{ if } \dim X = n < k$$

As previously mentioned, $\Omega^k(X)$ would be the zero vector space if $k > n$, and consequently both the space of closed and exact k-forms are zero. Therefore, the k-th de Rham cohomology group is zero modulo zero, which would just be zero.

Tools and Theorems

Poincaré Lemma

The Poincaré Lemma, in terms of cohomology, is a special case of homotopy invariance of de Rham cohomology for star-shaped open spaces. A star-shaped open space, also known as a star domain, is a region with at least one central point from which every other point within the space can be reached by a straight line segment that lies completely within the space. Star-shaped spaces are special cases of contractible spaces, so the Poincaré Lemma can be seen as a consequence of a more general result.

Theorem. If X is a contractible smooth manifold with or without boundary, then $H^k(X) = 0$ for $k \geq 1$.

Proof. If X is contractible, then it is homopoty equivalent to a point $p \in X$ by definition. So, $H^k(X) \cong H^k(p)$, and since $\dim p = 0$, its de Rham group of any positive degree is 0. □

Despite the apparent specialness of star-shaped domains, the Poincaré Lemma is one of the most important facts about de Rham cohomology.

Theorem. If U is a star-shaped open subset of $\mathbb{R}^n$ of $\mathbb{H}^n$, then $H^k(U) = 0$ for $k \geq 1$.

Proof. If U is star-shaped with respect to c, then it is contractible by the following straight-line homotopy:

$$H(x, t) = c + t(x - c)$$

Theorem 6.1.1 thus applies. □

Since $\mathbb{R}^n$ and $\mathbb{H}^n$ are both star-shaped, the following corollary holds.

Corollary. For any integers $n \geq 0$ and $k \geq 1$, $H^k(\mathbb{R}^n) = 0$ and $H^k(\mathbb{H}^n) = 0$.

The Mayer-Vietoris sequence

The Mayer-Vietoris sequence is a general theorem that can be used to compute the de Rham cohomology groups of many manifolds. To better understand how this sequence works, we begin with a brief refresher on some group theory concepts that will be useful in what follows.

Group Theory

Proposition. Let G, H be groups and $\varphi : G \to H$ be a group homomorphism. Let the identity element of G be 0_G. The map φ is injective if and only if $\ker \varphi = \{0_G\}$.

This shows the importance of the kernel in understanding the injectivity of a group homomorphism. More generally, the structure of the kernel and the image of a homomorphism are connected through the First Isomorphism Theorem, which is extremely useful for analyzing the structure of groups.

Theorem. If G, H are groups and $\varphi : G \to H$ is a group homomorphism, then there is an isomorphism between the quotient group $G/\ker \varphi$ and the image of φ. In other words,

$$\mathrm{im}\varphi \cong G/\ker \varphi$$

This theorem leads naturally to the Rank-Nullity Theorem in linear algebra. Since $G/\ker \varphi$ is isomorphic to the image of φ, their cardinalities can be equated. A vector space analogy of this property gives the following relation:

Theorem. Let V and W be finite-dimensional vector spaces and $T : V \to W$ a linear map. Then,

$$\dim V = \dim(\ker T) + \dim(\mathrm{im}T)$$

Exact Sequences

To define the Mayer-Vietoris sequence for de Rham cohomology, it is first necessary to introduce the notion of an exact sequence of vector spaces.

Definition. Consider a sequence of vector spaces and linear maps

$$\ldots \xrightarrow{f_{i-2}} A^{i-1} \xrightarrow{f_{i-1}} A^i \xrightarrow{f_i} A^{i+1} \xrightarrow{f_{i+1}} \ldots.$$

The sequence is said to be **exact at** A^i if

$$\mathrm{im}(f_{i-1}) = \ker(f_i).$$

Definition. A sequence is called an **exact sequence** if it is exact at every vector space in the sequence.

Definition. A **short exact sequence** is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

where 0 denotes the zero vector space. The exactness implies that f is injective, g is surjective, and $\mathrm{im}(f) = \ker(g)$.

Definition. A **long exact sequence** is an exact sequence that extends beyond three terms.

In many contexts, short exact sequences of complexes (chains of vector spaces with differentials) induce long exact sequences on cohomology groups.

Mayer-Vietoris Sequence

The Mayer-Vietoris Theorem gives a long exact sequence of de Rham cohomologies. The complete proof is relatively difficult, and uses the Snake Lemma, which will not be discussed here. The conclusion of the proof gives the following existence theorem:

Theorem. Let X be a smooth manifold with or without boundary, and let $U, V \subseteq X$ open such that $U \cup V = X$. For each k, there is a linear map $\delta : H^k(U \cap V) \to H^{k+1}(X)$ such that the following sequence, called the Mayer-Vietoris sequence, is exact:

$$\begin{aligned} \ldots \to H^{k-1}(U \cap V) &\xrightarrow{\delta_{k-1}} H^k(X) \xrightarrow{\zeta_k} H^k(U) \oplus H^k(V) \\ &\xrightarrow{\xi_k} H^k(U \cap V) \xrightarrow{\delta_k} H^{k+1}(X) \to \ldots \end{aligned}$$

The existence of this sequence allows the calculation of de Rham cohomology groups using the Rank-Nullity Theorem.

Example. $H^k(S^n)$ for all $k, n \geq 0$. Consider open subset $U, V \subset S^n$ as $U = S^n \setminus \{p\}$ and $V = S^n \setminus \{q\}$ for some points $p, q \in S^n$ where $p \neq q$. Then, U, V are homotopy equivalent to $\mathbb{R}^{n-1}$ and $U \cap V$ is homotopy equivalent to S^{n-1}. Applying Mayer-Vietoris, there must exist an exact sequence

$$\begin{aligned} 0 \xrightarrow{i} H^0(S^n) &\xrightarrow{\zeta_0} H^0(U) \oplus H^0(V) \xrightarrow{\xi_0} H^0(U \cap V) \xrightarrow{\delta_0} \\ H^1(S^n) &\xrightarrow{\zeta_1} H^1(U) \oplus H^1(V) \xrightarrow{\xi_1} H^1(U \cap V) \xrightarrow{\delta_1} \\ H^2(S^n) &\xrightarrow{\zeta_2} H^2(U) \oplus H^2(V) \xrightarrow{\xi_2} H^2(U \cap V) \xrightarrow{\delta_2} \ldots \end{aligned}$$

$$H^n(S^n) \xrightarrow{\zeta_n} H^n(U) \oplus H^n(V) \xrightarrow{\xi_n} H^n(U \cap V) \xrightarrow{\delta_n}$$
$$H^{n+1}(S^n) = 0 \to 0 \to \dots$$

By the connected pieces argument, $H^0(S^n) = \mathbb{R}$, $H^0(U) = \mathbb{R}$, $H^0(V) = \mathbb{R}$, and $H^0(U \cap V) = \mathbb{R}$ (unless $n = 1$ for which $H^0(U \cap V) = \mathbb{R}^2$). So, the sequence becomes

$$0 \xrightarrow{i} \mathbb{R} \xrightarrow{\zeta_0} \mathbb{R}^2 \xrightarrow{\xi_0} \mathbb{R}^2 \xrightarrow{\delta_0} H^1(S^n) \dots$$

for $n = 1$ and

$$0 \xrightarrow{i} \mathbb{R} \xrightarrow{\zeta_0} \mathbb{R}^2 \xrightarrow{\xi_0} \mathbb{R} \xrightarrow{\delta_0} H^1(S^n) \dots$$

otherwise. Now, $H^k(U) = H^k(\mathbb{R}^{n-1}) = H^k(V) = 0$ for all $k \geq 1$, so $H^k(U) \oplus H^k(V) = 0$ for all $k \geq 1$. So, the sequences become

$$0 \xrightarrow{i} \mathbb{R} \xrightarrow{\zeta_0} \mathbb{R}^2 \xrightarrow{\xi_0} \mathbb{R} \xrightarrow{\delta_0} H^1(S^n) \xrightarrow{\zeta_1} 0 \xrightarrow{\xi_1} H^1(U \cap V) \xrightarrow{\delta_1}$$
$$H^2(S^n) \xrightarrow{\zeta_2} 0 \xrightarrow{\xi_2} H^2(U \cap V) \xrightarrow{\delta_2} \dots$$
$$H^n(S^n) \xrightarrow{\zeta_n} 0 \xrightarrow{\xi_n} H^n(U \cap V) \xrightarrow{\delta_n} 0 \to \dots$$

where the map $\xi_0 : \mathbb{R}^2 \to \mathbb{R}^2$ for $n = 1$ and follows the diagram otherwise. Now, for all $k \geq 1$, δ_k is bijective due to the zero maps around it. Consider the following section of the sequence:

$$\dots \to 0 \xrightarrow{\xi_k} H^k(U \cap V) \xrightarrow{\delta_k} H^{k+1}(S^n) \xrightarrow{\zeta_{k+1}} 0 \to \dots$$

Since δ_k is bijective, $H^k(U \cap V) = H^{k+1}(S^n)$. Now, $U \cap V$ is homotopy equivalent to S^{n-1}, so $H^k(S^{n-1}) = H^{k+1}(S^n)$ for all $k \geq 1$. Thus, by induction, it is only necessary to find $H^1(S^n)$ for all n to find $H^k(S^n)$ for all k and n.
Now, the Rank-Nullity Theorem gives the following result for all $n \neq 1$:

$$\begin{aligned}
\text{im}(i) = 0 = \ker \zeta_0 \Rightarrow \dim(\ker \zeta_0) &= 0 \\
\dim(\ker \zeta_0) + \dim(\text{im}\zeta_0) = \dim(\mathbb{R}) &= 1 \\
\dim(\text{im}\zeta_0) &= 1 \\
\text{im}\zeta_0 = \mathbb{R} = \ker \xi_0 \Rightarrow \dim(\ker \xi_0) &= 1 \\
\dim(\ker \xi_0) + \dim(\text{im}\xi_0) = \dim(\mathbb{R}^2) &= 2 \\
\dim(\text{im}\xi_0) &= 1
\end{aligned}$$

$$\begin{aligned}
\text{im}\xi_0 = \mathbb{R} = \ker\delta_0 \Rightarrow \dim(\ker\delta_0) &= 1 \\
\dim(\ker\delta_0) + \dim(\text{im}\delta_0) = \dim(\mathbb{R}) &= 1 \\
\dim(\text{im}\delta_0) &= 0 \\
\text{im}\delta_0 = 0 = \ker\zeta_1 \Rightarrow \dim(\ker\zeta_1) &= 0 \\
\dim(\ker\zeta_1) + \dim(\text{im}\zeta_1) = \dim(H^1(S^n)) &
\end{aligned}$$

Now, $\xi_1 : 0 \to H^1(U \cap V)$ so its kernel is just 0. Since the sequence is exact, $\text{im}\zeta_0 = \ker\xi_1 = 0$, so $\dim(\text{im}\zeta_1) = 0$. So, $\dim(\ker\zeta_1) + \dim(\text{im}\zeta_1) = 0 + 0 = \dim(H^1(S^n)) \Rightarrow H^1(S^n) = 0$ for all $n \neq 1$. The same principle works for $n = 1$, but $\dim(\ker\delta_0) + \dim(\text{im}\delta_0) = \dim(\mathbb{R}^2) = 2$, so $\text{im}\delta_0 = \mathbb{R} = \ker\zeta_1$. The image of ζ_1 is still 0. So, $H^1(S^1) = \mathbb{R}$. Thus by induction,

$$H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

It is clear from this that the Mayer-Vietoris sequence is a powerful tool for calculating the de Rham Cohomology groups of manifolds.

Applications and Advanced Topics

The de Rham cohomology has many ties to other topological invariants. At a high level, many of these connections relate to the idea that cohomology measures the shape of a space by counting, in some algebraic sense, the number of independent k-dimensional "holes" it contains. Below is an exploration of two of these connections.

Relation to the Euler Characteristic

One important application of de Rham cohomology is its connection to the Euler characteristic of a manifold. This relationship is mediated through the Betti numbers, which are algebraic invariants encoding the ranks of the de Rham cohomology groups.

Definition. The **k-th Betti number** of a smooth manifold X is defined as

$$b_k = \dim H^k(X),$$

the dimension of the k-th de Rham cohomology group over X

These numbers reflect the number of k-dimensional "holes" in the manifold. The Euler–Poincaré Formula for Betti numbers gives that the alternating sum of the Betti numbers yields the Euler characteristic:

Theorem. Let X be a closed smooth manifold. Then the Euler characteristic $\chi(X)$ is given by

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k b_k.$$

Example. Consider the projective plane $\mathbb{RP}^2$. Mayer-Vietoris can be used to find the relation between dimensions of the de Rham cohomology groups. Consider an explicit open cover of $\mathbb{RP}^2$ using homogeneous coordinates. This allows us to set up a Mayer–Vietoris argument using charts. Define open subsets of $\mathbb{RP}^2$ as follows:

$$U = \{[x:y:z] \in \mathbb{RP}^2 | z \neq 0\}, V = \{[x:y:z] \in \mathbb{RP}^2 | x \neq 0 \text{ or } y \neq 0\}.$$

Then $U \cong \mathbb{R}^2$ via the chart $[x:y:z] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$, and $V \cong S^1$. The intersection $U \cap V \cong \mathbb{R}^2 \setminus \{0\} \simeq S^1$.
The Mayer–Vietoris sequence in de Rham cohomology for this open cover is

$$\begin{aligned} 0 \xrightarrow{i} & H^0(\mathbb{RP}^2) \xrightarrow{\zeta_0} H^0(U) \oplus H^0(V) \xrightarrow{\xi_0} H^0(U \cap V) \xrightarrow{\delta_0} \\ & H^1(\mathbb{RP}^2) \xrightarrow{\zeta_1} H^1(U) \oplus H^1(V) \xrightarrow{\xi_1} H^1(U \cap V) \xrightarrow{\delta_1} \\ & H^2(\mathbb{RP}^2) \xrightarrow{\zeta_2} H^2(U) \oplus H^2(V) \xrightarrow{\xi_2} H^2(U \cap V) \xrightarrow{\delta_2} \\ & H^3(\mathbb{RP}^2) = 0 \to 0 \to \dots \end{aligned}$$

Using the known cohomology of contractible spaces and spheres:

$$H^k(U) \cong H^k(V) \cong H^k(\mathbb{R}^2) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^k(U \cap V) \cong H^k(S^1) = \begin{cases} \mathbb{R} & k = 0, 1, \\ 0 & \text{otherwise}. \end{cases}$$

Substituting into the sequence gives

$$0 \xrightarrow{i} \mathbb{R} \xrightarrow{\zeta_0} \mathbb{R}^2 \xrightarrow{\xi_0} \mathbb{R} \xrightarrow{\delta_0} H^1(\mathbb{RP}^2) \xrightarrow{\zeta_1} \mathbb{R} \xrightarrow{\xi_1} \mathbb{R} \xrightarrow{\delta_1} H^2(\mathbb{RP}^2) \xrightarrow{\zeta_2} 0 \xrightarrow{\xi_2} \dots.$$

Now, using Rank-Nullity gives the following calculations:

$$\begin{aligned}
\mathrm{im}(i) = 0 = \ker\zeta_0 &\Rightarrow \dim(\ker\zeta_0) = 0\\
\dim(\ker\zeta_0) + \dim(\mathrm{im}\zeta_0) &= \dim(\mathbb{R}) = 1\\
\dim(\mathrm{im}\zeta_0) &= 1 = \dim(\ker\xi_0)\\
\dim(\ker\xi_0) + \dim(\mathrm{im}\xi_0) &= \dim(\mathbb{R}^2) = 2\\
\dim(\mathrm{im}\xi_0) &= 1 = \dim(\ker\delta_0)\\
\dim(\ker\delta_0) + \dim(\mathrm{im}\delta_0) &= \dim(\mathbb{R}) = 1\\
\dim(\mathrm{im}\delta_0) &= 0 = \dim(\ker\zeta_1)\\
\dim(\ker\zeta_1) + \dim(\mathrm{im}\zeta_1) &= \dim(H^1(\mathbb{RP}^2)) = b_1\\
\dim(\mathrm{im}\zeta_1) &= b_1 = \dim(\ker\xi_1)\\
\dim(\ker\xi_1) + \dim(\mathrm{im}\xi_1) &= \dim(\mathbb{R}) = 1\\
\dim(\mathrm{im}\xi_1) &= 1 - b_1 = \dim(\ker\delta_1)\\
\dim(\ker\delta_1) + \dim(\mathrm{im}\delta_1) &= \dim(\mathbb{R}) = 1\\
\dim(\mathrm{im}\delta_1) &= b_1 = \dim(\ker\zeta_2)\\
\dim(\ker\zeta_2) + \dim(\mathrm{im}\zeta_2) &= \dim(H^2(\mathbb{RP}^2)) = b_2\\
\dim(\mathrm{im}\zeta_2) &= b_2 - b_1 = \dim(\ker\xi_2)\\
\dim(\ker\xi_2) + \dim(\mathrm{im}\xi_2) &= \dim(0) = 0
\end{aligned}$$

Since ξ_2 maps 0 to 0, $\dim(\ker\xi_2) = \dim(\mathrm{im}\xi_2) = 0$. So, $b_2 - b_1 = 0$. Furthermore $b_0 = 1$ because $\mathbb{RP}^2$ has one connected component, and $b_k = 0$, $k \geq 3$ since the dimension of $\mathbb{RP}^2$ is 2, so all of its cohomology groups of degree greater than 2 must be zero. So, using Betti numbers, and the Euler characteristic is

$$\chi(\mathbb{R}P^2) = \sum_{k=0}^{\infty}(-1)^k b_k = 1 - b_1 + b_2 = 1 + 0 = 1.$$

Relation to Homotopy Groups

One powerful link between cohomology and homotopy theory lies in how the vanishing of homotopy groups constrains the structure of cohomology groups. In particular, there is a theorem that states:

Theorem. Let X be a path-connected topological space. If $\pi_n(X) = 0$ for all $1 \leq n \leq k$, then $H^k(X) = 0$.

Here, $\pi_n(X)$ denotes the nth homotopy group of the space X, which encodes information about the ways in which n-dimensional spheres S^n can be mapped continuously into X, up to homotopy. Specifically, $\pi_1(X)$ is the fundamental group, classifying loops in X up to deformation.For $n \geq 2$, $\pi_n(X)$ captures higher-dimensional analogs of this idea using n-spheres. The vanishing of all $\pi_n(X)$ for $n \leq k$ implies that X is k-connected—it has no "holes" up to dimension k. As a result, the kth cohomology group with integer coefficients must vanish: $H^k(X) = 0$. This shows the idea that if there are no nontrivial k-dimensional cycles (as detected by homotopy), then there can be no nontrivial k-dimensional cocycles (as detected by cohomology).

Conclusion

This paper has developed the theory of de Rham cohomology from the ground up, beginning with the geometry of smooth manifolds and differential forms, and culminating in the construction and application of cohomology groups. By exploring the algebraic properties of differential forms, including the wedge product, pullbacks, and exterior derivative, we found cohomology groups as algebraic invariants that capture global topological features of manifolds, especially the distinction between closed and exact forms.

The utility of de Rham cohomology was demonstrated through explicit computations, such as for spheres and projective spaces, using tools like the Poincaré Lemma and the Mayer–Vietoris sequence. Additionally, connections to other topological invariants such as Betti numbers, the Euler characteristic, and homotopy groups highlight the depth of de Rham cohomology in both differential and algebraic topology.

Ultimately, de Rham cohomology is a bridge between analysis and topology: it translates geometric intuition into algebraic structure. Through this, it not only provides computable invariants of manifolds but also deepens understanding of how local differential behavior informs global topological structure.[1][2] [3]

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